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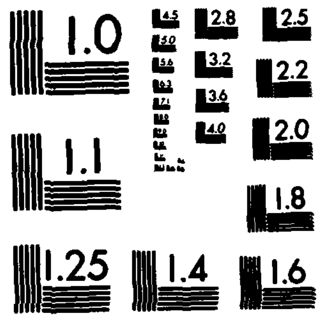

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A UNIQUENESS THEOREM IN ELASTODYNAMICS

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ABSTRACT

A uniqueness theorem is established for the scattering of harmonic elastic waves by a body with continuously varying parameters placed in a homogeneous medium.

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Key Words: linear elastic vibrations, uniqueness, inhomogeneous media.

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SIGNIFICANCE AND EXPLANATION

The passage of waves through a solid is a matter of some practical importance because of their effect on structures and also because they can be responsible for the transmission of noise. Vibration may occur naturally due to such thing as earth movement and wind gusting or may be caused by machinery. Sometimes oscillations are deliberately induced, as in ultrasonic testing, to check the strength of bonds, to detect flaws or to locate pockets of material different from their surroundings. Theoretical prediction of phenomena is therefore a vital adjunct of investigation in the field.

The theory is based on computing solutions of the equations of a mathematical model. These equations may, and usually do, have several solutions so there is a task of identification and interpretation of their relevance; clearly, wasted effort will be avoided if it can be indicated which solution or solutions to seek. Previously, it has been shown that, if a loss-free object of constant properties is embedded in another substance with constant material properties, there is only one solution which allows energy to radiate away from the body after it has been struck by an incoming wave. This report extends the theory to obstacles whose properties change continuously from point to point and also permits them to have losses. It shows that there is still a single solution to be found provided that one keeps to the mathematical criterion that the disturbance from the obstacle spreads outwards.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A UNIQUENESS THEOREM IN ELASTODYNAMICS

D. S. Jones

1. Introduction

In his book on elastodynamics Hudson (1980) points out the need for a uniqueness theorem for materials which are not homogeneous. This paper makes a contribution to filling that gap by providing a uniqueness theorem for inhomogeneous, but isotropic, bodies. Inhomogeneities of two types will be examined. Those in which the material parameters are constant except across certain surfaces of discontinuities and those in which the parameters vary from point to point but have some continuity and differentiability available. The piecewise homogeneous case has already been discussed by Kupradze (1963) but it is included here partly for completeness and partly because it involves only a short argument from formulae which are needed for the continuous inhomogeneity. In our investigation, the material parameters are permitted to be complex, with some limitations, so that lossy substances are not excluded from the theory. Only bonded bodies will be the subject of study.

Section 2 sets out the basic linear equations which are of concern and formulates some of the constraints on the field. Standard formulae for the representation of the displacement in a homogeneous body, whether finite or infinite, are given in Section 3.

As it turns out a proof of uniqueness for the general problem revolves about a proof for the interior of a finite body. This interior question is examined for the piecewise homogeneous substance in Section 4 and for the continuous inhomogeneity in Section 5. While the analysis of Section 4 is a straightforward application of the representation for the displacement that of

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Section 5 is much more complicated and probably the most difficult part of the whole exercise. Nevertheless, the theory here has no restrictions (other than continuity and differentiability) on the complex values of the material parameters though this freedom has to be abandoned in the full problem later. By means of the interior theorems the relevant theorems for the infinite medium are derived in Section 6 and their application to the uniqueness problem in scattering indicated in Section 7.

Two appendices contain results whose derivation would have interrupted the flow of argument in the main text. Appendix A gives, for easy reference, certain properties of spherical harmonics needed in Section 5. Appendix B covers the basic uniqueness theory and expansion properties in an infinite homogeneous medium. In particular, it verifies that one of the two customary radiation conditions can be disposed of without affecting uniqueness. Equations from the Appendices when referred to in the main text are distinguished by the appropriate letter.

2. The governing equations

The first problem to be considered is that of harmonic elastic waves in an isotropic body of finite size. The body occupies the volume T_- and its surface will be denoted by S. The volume outside S will be identified by T_+. In T_- the displacement at the point $\underset{\sim}{x}$ is $\underset{\sim}{u}(\underset{\sim}{x})$ with Cartesian components u_j and the stress tensor is τ_{jk}. The material occupying T_- is of density ρ and its elastic properties are specified by the Lamé parameters λ, μ. It will be assumed that there are no body forces. Then the equations to be satisfied in T_- are, when the time dependence is $e^{i\omega t}$,

$$\tau_{jk} = \lambda \frac{\partial u_m}{\partial x_m} \delta_{jk} + \mu\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) , \tag{1}$$

$$\frac{\partial\tau_{jk}}{\partial x_k} + \omega^2\rho u_j = 0 \tag{2}$$

where δ_{jk} is the standard Kronecker symbol and the usual summation convention has been employed. The Cartesian form of the governing equations is given with (x_1, x_2, x_3) the coordinates of the point $\underset{\sim}{x}$.

In most applications ρ is positive and, in lossfree media, $\lambda > 0$ and $\mu > 0$. When dissipation is present λ and μ can have imaginary parts. For much of the subsequent analysis they can be taken as arbitrary complex quantities but it will always be assumed that, on T_- and S,

$$|\mu| > 1/K, \; |\lambda+2\mu| > 1/K \quad , \tag{3}$$

where K is a positive finite constant.

The solutions of (1) and (2) to be found depend on the boundary conditions on S and whether there is any transfer of energy between T_- and T_+. When wave motion in T_+ has to be taken into account the media will be assumed to be bonded across the interface S so that the traction and displacement are continuous there.

3. The homogeneous medium

For subsequent purposes it will be convenient to have a representation of the displacement when the medium is homogeneous with λ, μ, ρ having the constant values λ_0, μ_0, ρ_0 respectively. It will be supposed that $\mu_0 \neq 0$ in concordance with (3). Let

$$g_{jk}(\underset{\sim}{x},\underset{\sim}{y}) = \frac{1}{4\pi\rho_0\omega^2}\frac{\partial^2}{\partial x_j \partial x_k}\left(\frac{e^{-i\omega a|\underset{\sim}{x}-\underset{\sim}{y}|} - e^{-i\omega b|\underset{\sim}{x}-\underset{\sim}{y}|}}{|\underset{\sim}{x}-\underset{\sim}{y}|}\right) - \frac{e^{-i\omega b|\underset{\sim}{x}-\underset{\sim}{y}|}}{4\pi\mu_0|\underset{\sim}{x}-\underset{\sim}{y}|}\delta_{jk} \tag{4}$$

where $a = \{\rho_0/(\lambda_0+2\mu_0)\}^{1/2}$, $b = (\rho_0/\mu_0)^{1/2}$ are the slownesses of P- and S-waves respectively. For complex parameters the square roots are defined by their principal values. In view of (3) both a and b are finite. The ranges of the parameters will be restricted so that

$$\mathrm{Im}(\omega a) \leq 0, \ \mathrm{Im}(\omega b) \leq 0 \tag{5}$$

but the real and imaginary parts of ωa are not permitted to vanish simultaneously nor are those of ωb. If $\mathrm{Im}(\omega a) = 0$ then ωa must be positive and similarly for ωb.

The tensor g_{jk} specified by (4) satisfies

$$(\lambda_0+\mu_0) \frac{\partial^2 g_{jk}}{\partial x_i \partial x_j} + \mu_0 \frac{\partial^2 g_{ik}}{\partial x_j \partial x_j} + \rho_0 \omega^2 g_{ik} = \delta_{ik}\delta(\underset{\sim}{x}-\underset{\sim}{y}) \quad . \tag{6}$$

By means of the divergence theorem and (1), (2), (6), Betti's representation (see, for example, Kupradze (1963)) can be obtained, namely

$$\begin{aligned} u_k(\underset{\sim}{x}) = \int_S \{ & \lambda_0 n_m u_m \frac{\partial}{\partial y_j} g_{jk}(\underset{\sim}{x},\underset{\sim}{y}) + \mu_0(n_i u_j + n_j u_i) \frac{\partial}{\partial y_i} g_{jk}(\underset{\sim}{x},\underset{\sim}{y}) \\ & - \tau_{ji} g_{jk}(\underset{\sim}{x},\underset{\sim}{y}) n_i \} ds_y \end{aligned} \tag{7}$$

for $\underset{\sim}{x}$ in T_-. In (7) $\underset{\sim}{n}$ is the unit normal to S out of T_- into T_+. If $\underset{\sim}{x}$ is in T_+ the left-hand side of (7) is replaced by zero.

Suppose now that T_+ is a homogeneous isotropic medium. A representation similar to (7) can be derived provided that appropriate radiation characteristics at infinity are prescribed. There are two possibilities that will be considered.

In the first case suppose that $\mathrm{Im}(\omega a) < 0$ and $\mathrm{Im}(\omega b) < 0$. It is then evident from (4) that g_{jk} decays exponentially at infinity. Assume that the displacement and stress also are exponentially attenuated at infinity. Then apply (7), which is valid for the interior of any closed surface, to the volume between a large spherical surface and the exterior side of S. Because of the exponential falling off of g_{jk}, $\underset{\sim}{u}$, τ_{jk} the contribution of the spherical surface tends to zero as the radius of the sphere approaches infinity. Therefore, if the definition of $\underset{\sim}{n}$ is retained unaltered, $u_k(\underset{\sim}{x})$ is the negative of the surface integral in (7) when $\underset{\sim}{x}$ is in T_+. Of course, it must be remembered that in this surface integral $\underset{\sim}{u}$ and τ_{jk} have the values on S which are approached from T_+ whereas when the integral represents u_k in T_- the values of $\underset{\sim}{u}$ and τ_{jk} on the interior side of S must be employed.

The second case is that in which $\mathrm{Im}(\omega a) = 0$ and $\mathrm{Im}(\omega b) = 0$. Now it is assumed that, as $R = |\underset{\sim}{x}| \to \infty$, $R\underset{\sim}{u}$ is bounded and that

$$R\{\hat{x}_m\tau_{jm} + i\omega b\mu_0(u_j - u_m\hat{x}_m\hat{x}_j) + i\omega a(\lambda_0+2\mu_0)\hat{x}_j u_m\hat{x}_m\} \to 0 \qquad (8)$$

where $\hat{\underset{\sim}{x}}$ is a unit vector in the direction of $\underset{\sim}{x}$. Actually, it is shown in Appendix B that the requirement for $R\underset{\sim}{u}$ to be bounded is superfluous but it is easier to justify some of the subsequent statements if this condition is retained. Bearing in mind that, as $|\underset{\sim}{x}| \to \infty$ with $\underset{\sim}{y}$ fixed,

$$g_{jk}(\underset{\sim}{x},\underset{\sim}{y}) \sim \frac{\hat{x}_j\hat{x}_k}{4\pi\rho_0 R}[b^2 e^{-i\omega b(R-\hat{\underset{\sim}{x}}\cdot\underset{\sim}{y})} - a^2 e^{-i\omega a(R-\hat{\underset{\sim}{x}}\cdot\underset{\sim}{y})}]$$
$$- \frac{\delta_{jk}}{4\pi\mu_0 R} e^{-i\omega b(R-\hat{\underset{\sim}{x}}\cdot\underset{\sim}{y})} \qquad (9)$$

we see that the contribution from the sphere at infinity again vanishes. The conclusion is that u_k can be represented in T_+ by the negative of the surface integral in (7).

It is conceivable that one of $\text{Im}(\omega a)$ and $\text{Im}(\omega b)$ could be zero while the other was negative but this does not seem to be of sufficient practical significance to justify separate consideration. Therefore, only the two cases already mentioned will be discussed subsequently.

4. The interior problem for piecewise homogeneity

This section and its successor are devoted to examining what can be said about the field in T_- when the displacement and traction vanish on S. Obviously, if the medium is homogeneous (7) may be invoked with the conclusion that $\underset{\sim}{u} \equiv \underset{\sim}{0}$ in T_-. It follows from (1) that the stress also disappears.

Actually (7) can be extended to piecewise homogeneous bodies in suitable circumstances. Let S_1 be a closed surface, entirely inside S, where the material properties change discontinuously from one set of constant values to another set. Apply (7) to the closed surface consisting of $S + S_1$. The integral over S is removed by the boundary conditions. As for the integral over S_1 it represents a field everywhere outside S_1 and it is obviously analytic. But it is identically zero outside S and therefore analytic continuation ensures that it is identically zero outside S_1. Therefore the traction and the displacement on the exterior of S_1 are zero. If the media are bonded across S_1 the traction and displacement are continuous and therefore zero on the inner side of S_1. If the interior of S_1 is homogeneous the preceding paragraph tells us that the field is zero. If there is another closed surface S_2 inside S_1 where the material properties change but in a bonded manner we repeat the argument starting from the closed surface $S_1 + S_2$. Obviously the procedure may be continued for any finite number of closed interfaces each totally enclosed by its predecessor. Such an arrangement may be distinguished by the adjective nested. There is, of course, no reason why S_1 should not consist of a finite number of distinct closed parts.

It must be emphasized that the above process may break down if, at any of the interfaces, the boundary conditions are other than the continuity of traction and displacement. In order to stress this point the word bonde' is included in the following theorem which has now been demonstrated.

Theorem 1. If the traction and displacement vanish on the bounded surface S of a bonded nested piecewise homogeneous body T_- the displacement and stress tensor are identically zero in T_-.

5. The interior problem for continuously inhomogeneous bodies

While Theorem 1 does cover certain inhomogeneous bodies the departure from homogeneity consists essentially of discontinuous changes across surfaces. In this section the material parameters will be assumed to vary from point to point but discontinuities will not be permitted. In fact, it will be assumed that ρ is continuous, λ is continuously differentiable and that μ is twice continuously differentiable.

Throughout solutions of (1) and (2) in T_- will be sought in which $\underset{\sim}{u}$ and $\partial u_j/\partial x_k$ are continuous. Furthermore, the boundary conditions

$$\underset{\sim}{u} = \underset{\sim}{0}, \; n_j \tau_{jk} = 0 \tag{10}$$

will be imposed on S.

If T_- were a piecewise homogeneous medium the boundary conditions (10) would be sufficient to ensure that the field vanished identically but the argument of the preceding section cannot be carried over to the case when the material parameters are continuously variable because the representation (7) is no longer available. Instead one is forced to proceed in a more indirect fashion.

In order to fix ideas the value zero is assigned to λ, μ, ρ in T_+.

Lemma 1. Let the field $\underset{\sim}{u}^{(1)}$, $\tau_{jk}^{(1)}$ be defined by

$$\underset{\sim}{u}^{(1)} = \underset{\sim}{u},\ \tau_{jk}^{(1)} = \tau_{jk} \qquad (\underset{\sim}{x} \in T_-) \quad ,$$

$$\underset{\sim}{u}^{(1)} = \underset{\sim}{0},\ \tau_{jk}^{(1)} = 0 \qquad (\underset{\sim}{x} \notin T_-) \quad .$$

Then, under the assumed conditions, $\underset{\sim}{u}^{(1)}$, $\partial u_j^{(1)}/\partial x_k$, $\tau_{jk}^{(1)}$, $\partial\tau_{jk}^{(1)}/\partial x_k$ are continuous everywhere.

Proof. Until the stated continuity has been established values on S when $\underset{\sim}{x}$ approaches there from T_- will be denoted by $(\ )_-$ and those for $\underset{\sim}{x}$ starting in T_+ will be signified by $(\ )_+$.

Continuity in T_+ is immediate because all quantities vanish and, moreover,

$$(\underset{\sim}{u}^{(1)})_+ = \underset{\sim}{0},\ (\partial u_j^{(1)}/\partial x_k)_+ = 0,\ (\tau_{jk}^{(1)})_+ = 0,\ (\partial\tau_{jk}^{(1)}/\partial x_k)_+ = 0 \quad . \tag{11}$$

In T_-, $\underset{\sim}{u}^{(1)}$ is continuous by assumption and (10) gives $(\underset{\sim}{u}^{(1)})_- = \underset{\sim}{0}$. Thus the continuity of $\underset{\sim}{u}^{(1)}$ has been verified. It is also clear from the hypotheses, (1) and (2) that $\partial u_j^{(1)}/\partial x_k$, $\tau_{jk}^{(1)}$ and $\partial\tau_{jk}^{(1)}/\partial x_k$ are continuous in T_-. Accordingly, the lemma will be confirmed once the requisite continuity across S has been demonstrated.

Choose any closed circuit C on S. Then, by Stokes's theorem,

$$\int_C (u_i^{(1)})_-\, d\underset{\sim}{s} = \int_{S'} \underset{\sim}{n} \wedge (\text{grad } u_i^{(1)})_-\, dS \tag{12}$$

where S' is the portion of S within C. By assumption (10), the left-hand side of (12) is zero. Since C is arbitrary, the conclusion is that $\underset{\sim}{n} \wedge (\text{grad } u_i^{(1)})_-$ is zero on S. A vector product with $\underset{\sim}{n}$ then enforces

$$\left(\frac{\partial u_i^{(1)}}{\partial x_j}\right)_- = n_j\left(\frac{\partial u_i^{(1)}}{\partial n}\right)_- \tag{13}$$

on S.

Also (10) says that $n_j(\tau_{jk}^{(1)})_- = 0$ on S and so, from (1),

$$\lambda n_k\left(\frac{\partial u_m^{(1)}}{\partial x_m}\right)_- + \mu n_j\left(\frac{\partial u_j^{(1)}}{\partial x_k} + \frac{\partial u_k^{(1)}}{\partial x_j}\right)_- = 0 \quad .$$

The insertion of (13) leads to

$$\lambda n_k n_m\left(\frac{\partial u_m^{(1)}}{\partial n}\right)_- + \mu n_j n_k\left(\frac{\partial u_j^{(1)}}{\partial n}\right)_- + \mu\left(\frac{\partial u_k^{(1)}}{\partial n}\right)_- = 0 \tag{14}$$

on S. Multiply (14) by n_k and then, by virtue of (3),

$$n_j(\partial u_j^{(1)}/\partial n)_- = 0 \quad . \tag{15}$$

On the other hand, if A_{ijk} is the alternate tensor, multiplication of (14) by $A_{pqk}n_q$ gives

$$A_{pqk}n_q(\partial u_k^{(1)}/\partial n)_- = 0 \tag{16}$$

on invoking (3) and noting that $A_{pqk}n_q n_k = 0$. Equations (15) and (16) may be expressed as $\underset{\sim}{n}\cdot\partial\underset{\sim}{u}^{(1)}/\partial n = 0$, $\underset{\sim}{n}\wedge\partial\underset{\sim}{u}^{(1)}/\partial n = \underset{\sim}{0}$ and therefore $(\partial\underset{\sim}{u}^{(1)}/\partial n)_- = 0$ on S. One infers from (13) that $(\partial u_i^{(1)}/\partial x_j)_- = 0$ on S. The required continuity of $\partial u_i^{(1)}/\partial x_j$ now follows from (11).

In view of (1) and (11) the continuity of $\tau_{jk}^{(1)}$ may be inferred whereas that of $\partial\tau_{jk}^{(1)}/\partial x_k$ is a consequence of (2), (11) and what has already been established about $\underset{\sim}{u}^{(1)}$. The lemma is proved.

Theorem 2. In T_- let $\underset{\sim}{u}$, $\partial u_j/\partial x_k$ be continuous and satisfy (1), (2) under the given assumptions. If $\underset{\sim}{u} = \underset{\sim}{0}$ and $n_j\tau_{jk} = 0$ on S then $\underset{\sim}{u} \equiv \underset{\sim}{0}$, $\tau_{jk} \equiv 0$ in T_-.

Proof. To permit analysis in the whole of space $\underset{\sim}{u}$ and τ_{jk} are first extended to $\underset{\sim}{u}^{(1)}$ and $\tau_{jk}^{(1)}$ as in Lemma 1. However, for simplicity of writing the affix (1) will be dropped, on the understanding that its presence is allowed for.

Pick as origin any point which can serve as the centre of some sphere within which the field vanishes identically. Clearly, there are many possible choices since any point of T_+ has the desired property.

Let R, θ, ϕ be spherical polar coordinates based on this origin. There are $2n + 1$ independent surface harmonics of order n. Construct an orthonormal set from them and let a typical member of that set be $S_{nj}(\theta,\phi)$ with $j = -n,-n+1,\ldots,n$. Then

$$\nabla^2\{R^n S_{nj}(\theta,\phi)\} = 0 \quad , \tag{17}$$

$$\nabla^2\{R^{-n-1} S_{nj}(\theta,\phi)\} = 0 \qquad (R > 0) \quad , \tag{18}$$

$$\int_\Omega S_{nj}S_{nk}\, d\Omega = \delta_{jk} \tag{19}$$

where Ω is the surface of the unit sphere.

Next, define

$$\psi_{nj}(\sigma,R) = \left(\frac{1}{R^{n+1}} - \frac{R^n}{\sigma^{2n+1}}\right)S_{nj}(\theta,\phi) \quad . \tag{20}$$

On account of (17) and (18)

$$\nabla^2\psi_{nj}(\sigma,R) = 0 \tag{21}$$

for $R > 0$. In addition $\psi_{nj}(\sigma,\sigma) = 0$.

The function ψ_{nj} has a singularity at the origin but, since it will always be multiplied by a field which is identically zero in a neighbourhood of the origin, the singularity can be effectively ignored in the subsequent analysis.

To abbreviate the notation the value of $\underset{\sim}{u}$ at the point R, θ, ϕ will be written as $\underset{\sim}{u}(R)$. Let $T(\sigma)$ be the interior of the sphere of radius σ and centre the origin. Then, by the divergence theorem,

$$\int_{T(\sigma)} \frac{\partial \tau_{ik}}{\partial x_k} \psi_{nj}(\sigma,R)d\underset{\sim}{x} = \int_\Omega n_k \tau_{ik}(\sigma)\psi_{nj}(\sigma,\sigma)\sigma^2 d\Omega$$

$$- \int_{T(\sigma)} \tau_{ik} \frac{\partial}{\partial x_k} \psi_{nj}(\sigma,R)d\underset{\sim}{x} \ .$$

It should be remarked that $T(\sigma)$ may encompass part or the whole of T_-. Therefore, possible discontinuities in the field across S have to be borne in mind in applying the divergence theorem to integrals over $T(\sigma)$. However, the continuity of τ_{jk} proved in Lemma 1 guarantees the validity of the above formula.

Since $\psi_{nj}(\sigma,\sigma)$ is zero

$$\int_{T(\sigma)} \frac{\partial \tau_{ik}}{\partial x_k} \psi_{nj}(\sigma,R)d\underset{\sim}{x} = -\int_{T(\sigma)} \{\lambda \frac{\partial u_m}{\partial x_m} \frac{\partial}{\partial x_i} \psi_{nj}(\sigma,R)$$

$$+ \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \frac{\partial}{\partial x_k} \psi_{nj}(\sigma,R)\}d\underset{\sim}{x} \qquad (22)$$

from (1). Now, by the divergence theorem,

$$\int_{T(\sigma)} \mu \frac{\partial u_i}{\partial x_k} \frac{\partial \psi_{nj}}{\partial x_k} d\underset{\sim}{x} = \int_\Omega n_k \mu u_i \frac{\partial \psi_{nj}}{\partial x_k} \sigma^2 d\Omega - \int_{T(\sigma)} u_i \frac{\partial \mu}{\partial x_k} \frac{\partial \psi_{nj}}{\partial x_k} d\underset{\sim}{x} \qquad (23)$$

on account of (21) and Lemma 1. Also

$$\int_{T(\sigma)} \mu \frac{\partial u_k}{\partial x_i} \frac{\partial \psi_{nj}}{\partial x_k} d\underset{\sim}{x} = \int_\Omega n_i \mu u_k \frac{\partial \psi_{nj}}{\partial x_k} \sigma^2 d\Omega - \int_{T(\sigma)} u_k \frac{\partial}{\partial x_i} (\mu \frac{\partial \psi_{nj}}{\partial x_k}) d\underset{\sim}{x}$$

$$= \int_\Omega \mu u_k \left(n_i \frac{\partial \psi_{nj}}{\partial x_k} - n_k \frac{\partial \psi_{nj}}{\partial x_i}\right) \sigma^2 d\Omega - \int_{T(\sigma)} u_k \frac{\partial \mu}{\partial x_i} \frac{\partial \psi_{nj}}{\partial x_k} d\underset{\sim}{x} \tag{24}$$

$$+ \int_{T(\sigma)} \frac{\partial}{\partial x_k} (\mu u_k) \frac{\partial \psi_{nj}}{\partial x_i} d\underset{\sim}{x}$$

after two applications of the divergence theorem and Lemma 1. Since ψ_{nj} vanishes when $R = \sigma$ advantage may be taken of (13) to assert that

$$\left[\frac{\partial \psi_{nj}}{\partial x_k}\right]_{R=\sigma} = n_k \left[\frac{\partial \psi_{nj}}{\partial R}\right]_{R=\sigma} = -n_k (2n+1) \frac{S_{nj}}{\sigma^{n+2}} . \tag{25}$$

Inserting (25) in (24) shows that there is no contribution from Ω. Hence, combining (22) - (25), we obtain

$$\frac{2n+1}{\sigma^n} \int_\Omega \mu u_i(\sigma) S_{nj} d\Omega = \int_{T(\sigma)} [\frac{\partial \tau_{ik}}{\partial x_k} \psi_{nj} + \{\lambda \frac{\partial u_m}{\partial x_m} + \frac{\partial}{\partial x_k} (\mu u_k)\} \frac{\partial \psi_{nj}}{\partial x_i}$$
$$- \{u_i \frac{\partial \mu}{\partial x_k} + u_k \frac{\partial \mu}{\partial x_i}\} \frac{\partial \psi_{nj}}{\partial x_k}] d\underset{\sim}{x} \tag{26}$$

wherein $\partial \tau_{ik}/\partial x_k$ may be replaced by $-\omega^2 \rho u_i$ by virtue of (2).

A further formula is helpful. It originates from

$$\int_{T(\sigma)} \frac{\partial \tau_{ik}}{\partial x_i} \frac{\partial \psi_{nj}}{\partial x_k} d\underset{\sim}{x} = \int_\Omega n_i \tau_{ik} \frac{\partial \psi_{nj}}{\partial x_k} \sigma^2 d\Omega - \int_{T(\sigma)} \tau_{ik} \frac{\partial^2 \psi_{nj}}{\partial x_i \partial x_k} d\underset{\sim}{x} \tag{27}$$

after drawing on Lemma 1 again. Now, from (21) and the symmetry of the double partial derivative,

$$\int_{T(\sigma)} \tau_{ik} \frac{\partial^2\psi_{nj}}{\partial x_i \partial x_k} d\underset{\sim}{x} = \int_{T(\sigma)} 2\mu \frac{\partial u_i}{\partial x_k} \frac{\partial^2\psi_{nj}}{\partial x_i \partial x_k} d\underset{\sim}{x}$$

$$= \int_\Omega 2n_k \mu u_i \frac{\partial^2\psi_{nj}}{\partial x_i \partial x_k} \sigma^2 d\Omega - \int_{T(\sigma)} 2u_i \frac{\partial \mu}{\partial x_k} \frac{\partial^2\psi_{nj}}{\partial x_i \partial x_k} d\underset{\sim}{x}$$

$$= \int_\Omega 2\Big(n_k \mu \frac{\partial^2\psi_{nj}}{\partial x_i \partial x_k} - n_i \frac{\partial \mu}{\partial x_k} \frac{\partial \psi_{nj}}{\partial x_k}\Big) u_i \sigma^2 d\Omega$$

$$+ \int_{T(\sigma)} 2 \frac{\partial}{\partial x_i} \Big(u_i \frac{\partial \mu}{\partial x_k}\Big) \frac{\partial \psi_{nj}}{\partial x_k} d\underset{\sim}{x} \quad . \tag{28}$$

Also

$$\int_\Omega n_i \tau_{ik} \frac{\partial \psi_{nj}}{\partial x_k} d\Omega = \int_\Omega \{n_k \lambda \frac{\partial u_m}{\partial x_m} + \mu n_i \Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big)\} \frac{\partial \psi_{nj}}{\partial x_k} d\Omega \tag{29}$$

and in here we can put

$$n_i \frac{\partial u_k}{\partial x_i} \frac{\partial \psi_{nj}}{\partial x_k} = n_k \frac{\partial u_k}{\partial x_i} \frac{\partial \psi_{nj}}{\partial x_i}$$

through (25). Hence (27) - (29) supply

$$\int_{T(\sigma)} \frac{\partial \tau_{ik}}{\partial x_i} \frac{\partial \psi_{nj}}{\partial x_k} d\underset{\sim}{x} = \int_\Omega \{\lambda \frac{\partial u_m}{\partial x_m} + 2 \frac{\partial}{\partial x_i} (\mu u_i)\} n_k \frac{\partial \psi_{nj}}{\partial x_k} \sigma^2 d\Omega$$

$$+ 2\int_\Omega \{n_i \frac{\partial}{\partial x_k} (\mu u_i) \frac{\partial \psi_{nj}}{\partial x_k} - n_k \frac{\partial}{\partial x_i} \Big(\mu u_i \frac{\partial \psi_{nj}}{\partial x_k}\Big)\} \sigma^2 d\Omega$$

$$- \int_{T(\sigma)} 2 \frac{\partial}{\partial x_i} \Big(u_i \frac{\partial \mu}{\partial x_k}\Big) \frac{\partial \psi_{nj}}{\partial x_k} d\underset{\sim}{x} \quad .$$

Since $\partial^2\psi_{nj}/\partial x_k \partial x_k = 0$ the integrand of the second surface integral can be expressed as

$$\Big(n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i}\Big) \mu u_i \frac{\partial \psi_{nj}}{\partial x_k} \quad .$$

But, if in (12) C is allowed to contract down to a point in such a way that S' becomes all of S, we see that the integral of $\underset{\sim}{n} \wedge \text{grad}\ \zeta$ over a closed surface is zero. Hence the second integral over Ω disappears and

$$\frac{2n+1}{\sigma^n} \int_\Omega \{\lambda \frac{\partial u_m}{\partial x_m}(\sigma) + 2\frac{\partial}{\partial x_i}(\mu u_i)\} S_{nj}\, d\Omega$$
$$= \int_{T(\sigma)} \{\rho\omega^2 u_k - 2\frac{\partial}{\partial x_i}\left(u_i \frac{\partial \mu}{\partial x_k}\right)\} \frac{\partial \psi_{nj}}{\partial x_k}\, d\underset{\sim}{x} \quad . \tag{30}$$

Note that in obtaining (26) and (30) no derivative or combinations of derivatives of u_i other than those occurring in Lemma 1 have been called for.

A rewriting of (26) and (30) is desirable. Put

$$\underset{\sim}{p}_{nj}(\sigma) = \int_\Omega \mu \underset{\sim}{u}(\sigma) S_{nj}\, d\Omega \quad ,$$

$$\underset{\sim}{q}_{nj}(\sigma) = \int_\Omega \rho\omega^2 \underset{\sim}{u}(\sigma) S_{nj}\, d\Omega \quad ,$$

$$r_{nj}(\sigma) = \int_\Omega \lambda \frac{\partial}{\partial x_m} u_m(\sigma) S_{nj}\, d\Omega \quad ,$$

$$s_{nj}(\sigma) = \int_\Omega \frac{\partial}{\partial x_k}(\mu u_k) S_{nj}\, d\Omega \quad ,$$

$$\underset{\sim}{t}_{njk}(\sigma) = \int_\Omega (\underset{\sim}{u} \frac{\partial \mu}{\partial x_k} + u_k \operatorname{grad} \mu) S_{nj}\, d\Omega \quad ,$$

$$\underset{\sim}{v}_{nj}(\sigma) = \int_\Omega \frac{\partial}{\partial x_i}(u_i \operatorname{grad} \mu) S_{nj}\, d\Omega \quad .$$

In addition (A.1) and (A.2) of Appendix A may be quoted to give when $n > 0$

$$\operatorname{grad} \psi_{nj} = \sum_{\alpha=-n-1}^{n+1} \underset{\sim}{b}_{\alpha j} S_{n+1,\alpha}/R^{n+2} - R^{n-1} \sum_{\beta=-n+1}^{n-1} \underset{\sim}{a}_{\beta j} S_{n-1,\beta}/\sigma^{2n+1}$$

where the summation convention does not apply to a repeated Greek suffix. Then, if $n > 0$, (26) furnishes

$$\frac{2n+1}{\sigma^n}\underset{\sim}{p}_{nj}(\sigma) = \int_0^\sigma[-(\frac{1}{R^{n-1}} - \frac{R^{n+2}}{\sigma^{2n+1}})\underset{\sim}{q}_{nj}(R) + \frac{1}{R^n}\sum_{\alpha=-n-1}^{n+1} \underset{\sim}{b}_{\alpha j}\{r_{n+1,\alpha}(R) + s_{n+1,\alpha}(R)\}$$

$$- \frac{R^{n+1}}{\sigma^{2n+1}}\sum_{\beta=-n+1}^{n-1} \underset{\sim}{a}_{\beta j}\{r_{n-1,\beta}(R) + s_{n-1,\beta}(R)\} \tag{31}$$

$$- \frac{1}{R^n}\sum_{\alpha=-n-1}^{n+1} (\underset{\sim}{b}_{\alpha j})_k \underset{\sim}{t}_{n+1,\alpha\, k}(R) + \frac{R^{n+1}}{\sigma^{2n+1}}\sum_{\beta=-n+1}^{n-1} (\underset{\sim}{a}_{\beta j})_k \underset{\sim}{t}_{n-1,\beta\, k}(R)]dR$$

while (30) provides

$$\frac{2n+1}{\sigma^n}\{r_{nj}(\sigma) + 2s_{nj}(\sigma)\} = \int_0^\sigma[\frac{1}{R^n}\sum_{\alpha=-n-1}^{n+1} \underset{\sim}{b}_{\alpha j}\cdot\{\underset{\sim}{q}_{n+1,\alpha}(R) - 2\underset{\sim}{v}_{n+1,\alpha}(R)\}$$

$$- \frac{R^{n+1}}{\sigma^{2n+1}}\sum_{\beta=-n+1}^{n-1} \underset{\sim}{a}_{\beta j}\cdot\{\underset{\sim}{q}_{n-1,\beta}(R) - 2\underset{\sim}{v}_{n-1,\beta}(R)\}]dR \tag{32}$$

for $j = -n, -n+1, \ldots, n$. If $n = 0$ (which entails $j = 0$) the terms involving $\underset{\sim}{a}_{\beta j}$ are missing because

$$\text{grad}\,\psi_{00} = \sum_{\alpha=-1}^{1} \underset{\sim}{b}_{\alpha j} S_{1,\alpha}/R^2 \quad .$$

With the convention that the terms in $\underset{\sim}{a}_{\beta j}$ are to be removed when n is placed equal to zero, (31) and (32) can be retained for all $n > 0$.

The notation will be further simplified by putting (31) and (32) in the form

$$\underset{\sim}{p}_{nj}(\sigma) = \frac{\sigma^n}{2n+1}\int_0^\sigma\{\frac{\underset{\sim}{A}_{nj}(R)}{R^n} + \frac{R^{n+1}}{\sigma^{2n+1}}\underset{\sim}{B}_{nj}(R)\}dR \quad , \tag{33}$$

$$r_{nj}(\sigma) + 2s_{nj}(\sigma) = \frac{\sigma^n}{2n+1}\int_0^\sigma\{\frac{C_{nj}(R)}{R^n} + \frac{R^{n+1}}{\sigma^{2n+1}}D_{nj}(R)\}dR \tag{34}$$

where

$$\underset{\sim}{A}_{nj}(R) = -R\underset{\sim}{q}_{nj} + \sum_{\alpha=-n-1}^{n+1}\{\underset{\sim}{b}_{\alpha j}(r_{n+1,\alpha} + s_{n+1,\alpha}) - (\underset{\sim}{b}_{\alpha j})_k \underset{\sim}{t}_{n+1,\alpha k}\} \quad , \tag{35}$$

$$\underset{\sim}{B}_{nj}(R) = R\underset{\sim}{q}_{nj} - \sum_{\beta=-n+1}^{n-1}\{\underset{\sim}{a}_{\beta j}(r_{n-1,\beta} + s_{n-1,\beta}) - (\underset{\sim}{a}_{\beta j})_k \underset{\sim}{t}_{n-1,\beta k}\} \quad , \tag{36}$$

$$C_{nj}(R) = \sum_{\alpha=-n-1}^{n+1} \underset{\sim}{b}_{\alpha j}\cdot(\underset{\sim}{q}_{n+1,\alpha} - 2\underset{\sim}{v}_{n+1,\alpha}) \quad , \tag{37}$$

$$D_{nj}(R) = -\sum_{\beta=-n+1}^{n-1} \underset{\sim}{a}_{\beta j}\cdot(\underset{\sim}{q}_{n-1,\beta} - 2\underset{\sim}{v}_{n-1,\beta}) \quad . \tag{38}$$

The body is of finite size and therefore there must exist positive finite $\sigma_1, \sigma_2 (\sigma_2 > \sigma_1)$ such that the field under consideration is identically zero inside the sphere of radius σ_1 and outside the sphere of radius σ_2, the centres of both spheres being at the origin. Obviously, σ_1 can be reduced and σ_2 increased, if convenient, so long as both remain finite and positive.

Since $\underset{\sim}{p}_{nj}(\sigma) = 0$ for $\sigma > \sigma_2$ the right-hand side of (33) must vanish for all such σ. However, the integrand also disappears for $R > \sigma_2$ and accordingly

$$\int_0^{\sigma_2} \{\underset{\sim}{A}_{nj}(R)/R^n\} dR = \underset{\sim}{0}, \; \int_0^{\sigma_2} R^{n+1} \underset{\sim}{B}_{nj}(R) dR = \underset{\sim}{0} \quad . \tag{39}$$

Similarly

$$\int_0^{\sigma_2} \{C_{nj}(R)/R^n\} dR = 0, \; \int_0^{\sigma_2} R^{n+1} D_{nj}(R) dR = 0 \quad . \tag{40}$$

Let p be a positive integer. Suppose firstly that $p > n$. Then, from (33),

$$\begin{aligned} |\underset{\sim}{p}_{nj}(\sigma)|^2 &< \frac{2\sigma^{2n}}{(2n+1)^2} \{ |\int_0^\sigma \frac{\underset{\sim}{A}_{nj}(R)}{R^n} dR|^2 + |\int_0^\sigma \frac{R^{n+1}\underset{\sim}{B}_{nj}(R)}{\sigma^{2n+1}} dR|^2 \} \\ &< \frac{2\sigma^{2n}}{(2n+1)^2} \{ \int_0^\sigma R^{2p-2n-\frac{1}{2}} dR \int_0^\sigma R^{\frac{1}{2}-2p} |\underset{\sim}{A}_{nj}(R)|^2 dR \\ &\quad + \int_0^\sigma R^{2n+2p+\frac{3}{2}} dR \int_0^\sigma R^{\frac{1}{2}-2p} \sigma^{-4n-2} |\underset{\sim}{B}_{nj}(R)|^2 dR \} \end{aligned}$$

by Schwarz's inequality. Hence

$$|\underset{\sim}{p}_{nj}(\sigma)|^2 < 2\sigma^{2p+\frac{1}{2}}\{2\int_0^\sigma R^{\frac{1}{2}-2p}|\underset{\sim}{A}_{nj}(R)|^2 dR$$

$$+\int_0^\sigma R^{\frac{1}{2}-2p}|\underset{\sim}{B}_{nj}(R)|^2 dR\}(2n+1)^{-2} \tag{41}$$

for $p > n$. When $1 < p < n$ we draw benefit from (39) and write

$$\left|\int_0^\sigma \frac{\underset{\sim}{A}_{nj}(R)}{R^n}\,dR\right|^2 = \left|\int_\sigma^{\sigma_2} \frac{\underset{\sim}{A}_{nj}(R)}{R^n}\,dR\right|^2 < \int_\sigma^{\sigma_2} R^{2p+\frac{3}{4}-2n}dR \int_\sigma^{\sigma_2} R^{-2p-\frac{3}{4}}|\underset{\sim}{A}_{nj}(R)|^2 dR$$

for $\sigma < \sigma_2$. Since

$$\int_\sigma^{\sigma_2} R^{2p+\frac{3}{4}-2n}dR < \int_\sigma^\infty R^{2p+\frac{3}{4}-2n}dR$$

we deduce that

$$(2n+1)^2|\underset{\sim}{p}_{nj}(\sigma)|^2 < 8\sigma^{2p+\frac{7}{4}}\int_\sigma^{\sigma_2} R^{-2p-\frac{3}{4}}|\underset{\sim}{A}_{nj}(R)|^2 dR$$

$$+ 2\sigma^{2p+\frac{1}{2}}\int_0^\sigma R^{\frac{1}{2}-2p}|\underset{\sim}{B}_{nj}(R)|^2 dR \tag{42}$$

for $1 < p < n$ and $\sigma < \sigma_2$.

From (41) and (42) follows

$$\sum_{n=0}^\infty \sum_{j=-n}^n |\underset{\sim}{p}_{nj}(\sigma)|^2 < \sum_{n=0}^p \sum_{j=-n}^n 4\sigma^{2p+\frac{1}{2}}\int_0^\sigma R^{\frac{1}{2}-2p}|\underset{\sim}{A}_{nj}(R)|^2 dR(2n+1)^{-2}$$

$$+\sum_{n=p+1}^\infty \sum_{j=-n}^n 8\sigma^{2p+\frac{7}{4}}\int_\sigma^{\sigma_2} R^{-2p-\frac{3}{4}}|\underset{\sim}{A}_{nj}(R)|^2 dR(2n+1)^{-2} \tag{43}$$

$$+\sum_{n=0}^\infty \sum_{j=-n}^n 2\sigma^{2p+\frac{1}{2}}\int_0^\sigma R^{\frac{1}{2}-2p}|\underset{\sim}{B}_{nj}(R)|^2 dR(2n+1)^{-2}$$

for $\sigma < \sigma_2$.

To estimate the right-hand side of (43) observe from (35) that

$$|\underset{\sim}{A}_{nj}(R)|^2 < 3\sigma_2^2|\underset{\sim}{g}_{nj}|^2 + 3|\sum_{\alpha=-n-1}^{n+1} \underset{\sim}{b}_{\alpha j}(r_{n+1,\alpha} + s_{n+1,\alpha})|^2$$

$$+ 3|\sum_{\alpha=-n-1}^{n+1} (\underset{\sim}{b}_{\alpha j})_k \, \underset{\sim}{t}_{n+1,\alpha k}|^2 \; .$$

Now

$$\sum_{j=-n}^{n} |\sum_{\alpha=-n-1}^{n+1} \underset{\sim}{b}_{\alpha j}(r_{n+1,\alpha} + s_{n+1,\alpha})|^2 = \frac{(n+1)(2n+1)^2}{2n+3} \sum_{\alpha=-n-1}^{n+1} |r_{n+1,\alpha} + s_{n+1,\alpha}|^2 \tag{44}$$

from (A.8). Further

$$|\sum_{\alpha=-n-1}^{n+1} (\underset{\sim}{b}_{\alpha j})_k \, \underset{\sim}{t}_{n+1,\alpha k}|^2 < 9 \sum_{r=1}^{3} \sum_{s=1}^{3} |\sum_{\alpha=-n-1}^{n+1} \underset{\sim}{b}_{\alpha j}(\underset{\sim}{t}_{n+1,\alpha r})_s|^2$$

and so

$$\sum_{j=-n}^{n} |\sum_{\alpha=-n-1}^{n+1} (\underset{\sim}{b}_{\alpha j})_k \, \underset{\sim}{t}_{n+1,\alpha k}|^2 < \frac{9(n+1)(2n+1)^2}{2n+3} \sum_{s=1}^{3} \sum_{\alpha=-n-1}^{n+1} |\underset{\sim}{t}_{n+1,\alpha s}|^2$$

from (44). Hence

$$\sum_{j=-n}^{n} |\underset{\sim}{A}_{nj}(R)|^2 < 3\sigma_2^2 \sum_{j=-n}^{n} |\underset{\sim}{g}_{nj}|^2 + 3(2n+1)^2 \sum_{\alpha=-n-1}^{n+1} |r_{n+1,\alpha} + s_{n+1,\alpha}|^2$$

$$+ 27(2n+1)^2 \sum_{s=1}^{3} \sum_{\alpha=-n-1}^{n+1} |\underset{\sim}{t}_{n+1,\alpha s}|^2 \; .$$

Another observation is that

$$\sum_{n=0}^{\infty} \sum_{j=-n}^{n} |\underset{\sim}{g}_{nj}(R)|^2 = \int_\Omega |\rho\omega^2 \underset{\sim}{u}(R)|^2 d\Omega$$

on account of the completeness of the spherical harmonics. Since ρ is bounded it follows that

$$\sum_{n=0}^{\infty} \sum_{j=-n}^{n} |\underset{\sim}{g}_{nj}(R)|^2 < B \int_\Omega |\underset{\sim}{u}(R)|^2 d\Omega \quad .$$

Similarly

$$\sum_{n=0}^{\infty} \sum_{\alpha=-n-1}^{n+1} |r_{n+1,\alpha}(R) + s_{n+1,\alpha}(R)|^2 < B \int_\Omega \{|\underset{\sim}{u}(R)|^2 + |\partial u_m/\partial x_m|^2\} d\Omega \ ,$$

$$\sum_{n=0}^{\infty} \sum_{s=1}^{3} \sum_{\alpha=-n-1}^{n+1} |\underset{\sim}{t}_{n+1,\alpha s}|^2 < B \int_\Omega |\underset{\sim}{u}(R)|^2 d\Omega$$

where B is now being used generically for a constant independent of R.

It may be shown in a similar fashion that

$$\sum_{j=-n}^{n} |B_{nj}(R)|^2 < 3\sigma_2^2 \sum_{j=-n}^{n} |\underset{\sim}{g}_{nj}|^2 + 3(2n+1)^2 \sum_{\beta=-n+1}^{n-1} |r_{n-1,\beta} + s_{n-1,\beta}|^2$$

$$+ 27(2n+1)^2 \sum_{s=1}^{3} \sum_{\beta=-n+1}^{n-1} |\underset{\sim}{t}_{n-1,\beta s}|^2$$

the last two terms being absent when $n = 0$. With this convention

$$\sum_{n=0}^{\infty} \sum_{\beta=-n+1}^{n-1} |r_{n-1,\beta} + s_{n-1,\beta}|^2 < B \int_\Omega \{|\underset{\sim}{u}(R)|^2 + |\partial u_m/\partial x_m|^2\} d\Omega \ ,$$

$$\sum_{n=0}^{\infty} \sum_{s=1}^{3} \sum_{\beta=-n+1}^{n-1} |\underset{\sim}{t}_{n-1,\beta s}|^2 < B \int_\Omega |\underset{\sim}{u}(R)|^2 d\Omega \quad .$$

Let us now suppose that

$$\int_\Omega \{|\underset{\sim}{u}(R)|^2 + |\partial u_m/\partial x_m|^2\} d\Omega < C_p R^{2p-\frac{1}{2}} \quad . \tag{45}$$

This statement is obviously valid when $p = 1$ from the assumed properties of the field. Indeed, the left-hand side is zero when $R < \sigma_1$ and when

$R > \sigma_2$. Then (43) and the succeeding inequalities reveal that

$$\int_\Omega |\mu \underset{\sim}{u}(\sigma)|^2 d\Omega = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} |p_{jn}(\sigma)|^2 < B\, C_p \sigma^{2p+\frac{3}{2}}(1+\sigma_2^2)$$

for $\sigma < \sigma_2$. Invoking (3) we infer that

$$\int_\Omega |\underset{\sim}{u}(\sigma)|^2 d\Omega < BK^2 C_p \sigma^{2p+\frac{3}{2}}(1+\sigma_2^2) \quad . \tag{46}$$

A similar analysis applied to (34) furnishes

$$\int_\Omega |\lambda \frac{\partial}{\partial x_m} u_m(\sigma) + 2 \frac{\partial}{\partial x_k} (\mu u_k)|^2 d\Omega < B\, C_p \sigma^{2p+\frac{3}{2}} \tag{47}$$

and again (3), together with (46), gives

$$\int_\Omega \left|\frac{\partial u_m}{\partial x_m}\right|^2 < BK^2 C_p \sigma^{2p+\frac{3}{2}}(1+\sigma_2^2) \quad . \tag{48}$$

Inequalities (46) and (48) may be combined as

$$\int_\Omega \{|\underset{\sim}{u}(R)|^2 + |\partial u_m/\partial x_m|^2\} d\Omega < BK^2 C_p R^{2p+\frac{3}{2}}(1+\sigma_2^2) \tag{49}$$

for $R < \sigma_2$.

Then, when (45) holds for some p, (49) shows that it holds for $p + 1$ provided that $C_{p+1} = BK^2 C_p (1+\sigma_2^2)$. Since it is certainly true for $p = 1$

$$\int_\Omega \{|\underset{\sim}{u}(R)|^2 + |\partial u_m/\partial x_m|^2\} d\Omega < \{BK^2(1+\sigma_2^2)\}^{p-1} C_1 R^{2p-\frac{1}{2}}$$

for $R < \sigma_2$ and $p = 1,2,\ldots$. An immediate consequence of letting $p \to \infty$ is that $\underset{\sim}{u} \equiv \underset{\sim}{0}$ for $R < 1/B^{\frac{1}{2}}K(1+\sigma_2^2)^{\frac{1}{2}}$. It is transparent that $\tau_{jk} \equiv 0$ in the same region.

Let $R_0 = 1/B^{\frac{1}{2}}K(1+\sigma_2^2)^{\frac{1}{2}}$. Choose another origin $\underset{\sim}{x}_1$ which lies within the sphere of radius R_0. The body is entirely inside the original sphere of radius σ_2 and therefore contained in a sphere of radius $\sigma_2 + R_0$ about $\underset{\sim}{x}_1$. However, the field is identically zero in a neighbourhood of $\underset{\sim}{x}_1$ and so the foregoing theory may be applied to show that it is identically zero inside a sphere of radius

$$R_1 = \{1 + (\sigma_2+R_0)^2\}^{-\frac{1}{2}}/B^{\frac{1}{2}}K \quad .$$

In view of the arbitrary selection of $\underset{\sim}{x}_1$ we deduce that the field is identically zero throughout a sphere of radius $R_0 + R_1$ centered on the original origin. Clearly, the process can be continued and the field will be identically zero inside a sphere of radius $R_0 + R_1 + R_2 + \ldots + R_n = S_n$ where

$$R_n = \{1 + (S_{n-1} + R_0)^2\}^{-\frac{1}{2}}/B^{\frac{1}{2}}K \quad . \tag{50}$$

If S_n approached a finite limit as $n \to \infty$, it would be necessary for $R_n \to 0$ which is inconsistent with (50). Therefore, the body will be totally enclosed by the sphere of radius S_n after a finite number of steps. Thus the field is identically zero throughout the body and the theorem is proved.

6. The exterior problem

The next matter to be examined is what happens when the regions T_+ and T_- are bonded across S and, instead of specifying the displacement and traction on S, the behaviour at infinity is prescribed. It will be supposed that the material parameters λ, μ, ρ have in T_+ the constant values λ_0, μ_0, ρ_0 respectively. The restrictions that will be imposed are covered by Conditions A. In addition to satisfying (3) the material parameters must comply, in both T_- and T_+, with either

(a) $\mathrm{Im}(\mu) \geq 0$, $\mathrm{Im}(\lambda+2\mu) \geq 0$, $\mathrm{Im}(\omega^2\rho) \leq 0$, *or* (b) $\mathrm{Im}(\mu) > 0$, $\mathrm{Im}(\lambda+2\mu) > 0$, $\omega^2\rho > 0$,

or (c) $\mu > 0$, $\lambda + 2\mu > 0$, $\omega^2\rho > 0$.

There is no difficulty in checking that any one of the alternatives is sufficient to ensure the validity of the constraints placed on ωa and ωb in §3. Furthermore, under any of the three conditions,

$$\mathrm{Im}(i\omega b\mu_0) > 0, \ \mathrm{Im}\{i\omega a(\lambda_0 + 2\mu_0)\} > 0 \ . \tag{51}$$

We shall now prove the following theorem.

Theorem 3. Let ρ *be continuous,* λ *be continuously differentiable and* μ *twice continuously differentiable in* $T_- + S$ *and have the constant values* ρ_0, λ_0, μ_0 *in* T_+. *Let* $\underset{\sim}{u}, \tau_{jk}$ *satisfy (1) and (2) when* $\underset{\sim}{x} \notin S$ *and be such that* u_i *and* $\partial u_j/\partial x_k$ *are continuous in both* $T_- + S$ *and* $T_+ + S$. *Suppose further that*

$$(\underset{\sim}{u})_+ = (\underset{\sim}{u})_-, \ (n_j\tau_{jk})_+ = (n_j\tau_{jk})_- \tag{52}$$

on S. *Then, if* $\underset{\sim}{u}, \tau_{jk}$ *satisfy the radiation conditions (8) at infinity* $\underset{\sim}{u} \equiv \underset{\sim}{0}$, $\tau_{jk} \equiv 0$ *subject to Conditions A*.

Proof. Let R be so large that a sphere of radius R centered on the origin totally surrounds S. Then

$$\int_\Omega n_j\tau_{jk}(R)u_k^*(R)R^2 d\Omega = \int_S n_j(\tau_{jk}u_k^*)_+ dS$$
$$+ \int_T \{\lambda_0 \left|\frac{\partial u_m}{\partial x_m}\right|^2 + \frac{1}{2}\mu_0\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)\left(\frac{\partial u_j^*}{\partial x_k} + \frac{\partial u_k^*}{\partial x_j}\right) - \rho_0\omega^2|\underset{\sim}{u}|\}d\underset{\sim}{x}$$

where T is the volume between S and the surface of the sphere of radius R. The star signifies a complex conjugate. By virtue of (52), $n_j(\tau_{jk}u_k^*)_+$ may be replaced in the integral over S by $n_j(\tau_{jk}u_k^*)_-$ and then the divergence theorem applied to T_-. Hence

$$\int_\Omega n_j\tau_{jk}(R)u_k^*(R)R^2 d\Omega$$

$$= \int_{T(R)}\{\lambda\left|\frac{\partial u_m}{\partial x_m}\right|^2 + \frac{1}{2}\mu\left(\frac{\partial u_j}{\partial x_k}+\frac{\partial u_k}{\partial x_j}\right)\left(\frac{\partial u_j^*}{\partial x_k}+\frac{\partial u_k^*}{\partial x_j}\right) - \rho\omega^2|\underset{\sim}{u}|^2\}d\underset{\sim}{x} \quad .$$

The imposition of the radiation conditions (8) now enables one to say that, as $R \to \infty$,

$$\int_{T(R)}[(\lambda+2\mu)\left|\frac{\partial u_m}{\partial x_m}\right|^2 + \mu\{\left|\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right|^2 + \left|\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right|^2$$

$$+ \left|\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right|^2\} - \rho\omega^2|\underset{\sim}{u}|^2]d\underset{\sim}{x} = -\int_\Omega\{i\omega b\mu_0(|\underset{\sim}{u}|^2 - |\hat{\underset{\sim}{x}}\cdot\underset{\sim}{u}|^2) \qquad (53)$$

$$+ i\omega a(\lambda_0+2\mu_0)|\hat{\underset{\sim}{x}}\cdot\underset{\sim}{u}|^2\}R^2 d\Omega + o(1) \quad .$$

Under conditions A the imaginary part of the left-hand side of (53) cannot be negative whereas, on account of (51), the imaginary part of the right-hand side cannot be positive. This is possible only if

$$\lim_{R\to\infty}\int_{T(R)}[(\lambda+2\mu)\left|\frac{\partial u_m}{\partial x_m}\right|^2 + \mu\{\left|\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right|^2 + \left|\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right|^2 + \left|\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right|^2\} \qquad (54)$$

$$- \rho\omega^2|\underset{\sim}{u}|^2]d\underset{\sim}{x} = 0 \quad ,$$

$$\lim_{R\to\infty}\int_\Omega\{b\mu_0(|\underset{\sim}{u}|^2 - |\hat{\underset{\sim}{x}}\cdot\underset{\sim}{u}|^2) + a(\lambda_0+2\mu_0)|\hat{\underset{\sim}{x}}\cdot u|^2\}R^2 d\Omega = 0 \quad . \qquad (55)$$

It is evident at once from (54) that, for conditions A(a), $\underset{\sim}{u} \equiv \underset{\sim}{0}$. It follows that $\tau_{jk} \equiv 0$ and the theorem is proved. Similarly, under A(b), (54) implies that $\tau_{jk} \equiv 0$ and then (2) enforces $\underset{\sim}{u} \equiv \underset{\sim}{0}$. Again the theorem has been demonstrated.

For conditions A(c), (54) is no longer helpful and (55) must be turned to. In Appendix B it is shown that the radiation conditions entail (equation (B.11))

$$\underset{\sim}{u} = -i\omega a\, f_0 \frac{e^{-i\omega aR}}{R} \hat{\underset{\sim}{x}} - i\omega b \hat{\underset{\sim}{x}} \wedge \underset{\sim}{g}_0 \frac{e^{-i\omega bR}}{R} + O(\frac{1}{R^2})$$

subject to

$$\underset{\sim}{g}_0 \cdot \hat{\underset{\sim}{x}} = 0 \quad . \tag{56}$$

Since $\hat{\underset{\sim}{x}}$ and $\hat{\underset{\sim}{x}} \wedge \underset{\sim}{g}_0$ are perpendicular the implication of (56) is that

$$|\underset{\sim}{u}|^2 = \{(\omega a|f_0|)^2 + (\omega b|\underset{\sim}{g}_0|)^2\}/R^2 + O(1/R^3) \quad .$$

Accordingly,

$$\int_\Omega \{\mu_0 \omega^2 b^3 |\underset{\sim}{g}_0|^2 + (\lambda_0 + 2\mu_0)\omega^2 a^3 |f_0|^2\} d\Omega = 0$$

stems from (55).

By hypothesis the coefficient of $|\underset{\sim}{g}_0|^2$ is non-zero and so we are obliged to have $\underset{\sim}{g}_0 \equiv \underset{\sim}{0}$. Similarly, $f_0 \equiv 0$. But, in Appendix B, it has been demonstrated that $f_0 \equiv 0$ and $\underset{\sim}{g}_0 \equiv \underset{\sim}{0}$ make $\underset{\sim}{u} \equiv \underset{\sim}{0}$ in T_+. Hence $\tau_{jk} \equiv 0$ in T_+. It follows that $(\underset{\sim}{u})_+ = \underset{\sim}{0}$ and $(n_j \tau_{jk})_+ = 0$ on S. From (52), $(\underset{\sim}{u})_- = \underset{\sim}{0}$ and $(n_j \tau_{jk})_- = 0$ and now Theorem 2 enforces $\underset{\sim}{u} \equiv \underset{\sim}{0}$ and $\tau_{jk} \equiv 0$ in T_-. The proof of the theorem is finished.

Evidently, the same method of proof but drawing on Theorem 1 can be applied to the piecewise homogeneous body and so we can state

Theorem 4. Let T_- be a bonded nested piecewise homogeneous body. Let $\underset{\sim}{u}$, τ_{jk} satisfy (1) and (2) except on any interface and have continuity properties analogous to those of Theorem 3. If $\underset{\sim}{u}$, τ_{jk} comply with the bonded boundary condition and satisfy the radiation conditions at infinity then, under conditions A, $\underset{\sim}{u} \equiv \underset{\sim}{0}$ and $\tau_{jk} \equiv 0$.

The theorems have been proved with the boundedness of $R|\underset{\sim}{u}|$ as part of the radiation conditions. However, it is demonstrated at the end of Appendix B that this requirement can be dropped without modifying the assertion that $f_0 \equiv 0$, $\underset{\sim}{g}_0 \equiv \underset{\sim}{0}$ implies that $\underset{\sim}{u} \equiv \underset{\sim}{0}$ in T_+. Therefore, the validity of Theorems 3 and 4 is unaffected by this change to the radiation conditions. We state this as a corollary.

Corollary. Theorems 3 and 4 remain true when (8) is the sole radiation condition.

7. Uniqueness for scattering

An easy consequence of the theorems of the preceding section is that the solution to the scattering problem is unique. Let an incident displacement $\underset{\sim}{u}^i$ be generated by some means. It is presumed that any scattered field produced is outgoing at infinity and so satisfies the radiation conditions. If there were two possible scattered fields, taking the difference between the total fields would eliminate $\underset{\sim}{u}^i$ and have a field satisfying our theorems. That field must therefore be identically zero and uniqueness of the scattered wave has been established.

Theorem 5. If a given incident wave produces a scattered wave satisfying the radiation condition then, under the conditions of either Theorem 3 or 4, the scattered wave is unique.

It is, of course, sufficient for uniqueness to impose the radiation condition in the form (8) alone but it may be more convenient, in practice, to keep the boundedness of $R\underset{\sim}{u}$ available even though it is jettisonable.

APPENDIX A

This appendix is devoted to deriving some properties of spherical harmonics which are needed in the main text. The summation convention is not employed in this appendix.

Since $R^n S_{nj}(\theta,\phi)$ is a solution of Laplace's equation so is $\partial(R^n S_{nj})/\partial x_k$. It will, however, be of one degree lower and hence is expressible in terms of solid harmonics of order $n - 1$. Hence there are constant vectors $\underset{\sim}{a}_{pj}$ such that

$$\mathrm{grad}(R^n S_{nj}) = R^{n-1} \sum_{p=-n+1}^{n-1} \underset{\sim}{a}_{pj} S_{n-1,p} \quad (j = -n,-n+1,\ldots,n) \quad . \tag{A.1}$$

A similar argument reveals that there are constant vectors $\underset{\sim}{b}_{qk}$ such that

$$\mathrm{grad}(S_{n-1,k}/R^n) = \sum_{q=-n}^{n} \underset{\sim}{b}_{qk} S_{nq}/R^{n+1} (k = -n+1,\ldots,n-1) \quad . \tag{A.2}$$

It is understood that $n \geq 1$ in (A.1) since S_{00} is constant and $\mathrm{grad}\, S_{00} = \underset{\sim}{0}$.

From (A.1)

$$\begin{aligned}\mathrm{div}\{R^{1-2n}\mathrm{grad}\,(R^n S_{nj})\} &= \sum_{p=-n+1}^{n-1} \underset{\sim}{a}_{pj} \cdot \mathrm{grad}(S_{n-1,p}/R^n) \\ &= \sum_{p=-n+1}^{n-1} \underset{\sim}{a}_{pj} \cdot \sum_{q=-n}^{n} \underset{\sim}{b}_{qp} S_{nq}/R^{n+1}\end{aligned}$$

on account of (A.2). Since $\nabla^2(R^n S_{nj}) = 0$ we deduce that

$$(1-2n)nS_{nj} = \sum_{p=-n+1}^{n-1} \underset{\sim}{a}_{pj} \cdot \sum_{q=-n}^{n} \underset{\sim}{b}_{qp} S_{nq} \quad .$$

The orthonormal property of the S_{nj} now leads to the conclusion

$$\sum_{p=-n+1}^{n-1} \underset{\sim}{a}_{pj} \cdot \underset{\sim}{b}_{qp} = (1-2n)n\delta_{qj} \quad (q,j = -n,\ldots,n) \quad . \tag{A.3}$$

Similarly, by taking the divergence of (A.2) multiplied by R^{2n+1} we infer that

$$\sum_{q=-n}^{n} \underset{\sim}{b}_{qk} \cdot \underset{\sim}{a}_{pq} = (1+2n)n\delta_{kp} \quad (k,p = -n+1,\ldots,n-1) \quad . \tag{A.4}$$

Next, form the scalar produce of (A.1) and itself with j replaced by q. Then

$$R^{2n-2} \sum_{p=-n+1}^{n-1} \underset{\sim}{a}_{pj} S_{n-1,p} \cdot \sum_{r=-n+1}^{n-1} \underset{\sim}{a}_{rq} S_{n-1,r} = \text{grad}(R^n S_{nj}) \cdot \text{grad}(R^n S_{nq})$$

whence

$$R^{2n-2} \sum_{p=-n+1}^{n-1} \underset{\sim}{a}_{pj} \cdot \underset{\sim}{a}_{pq} = \int_\Omega \text{grad}(R^n S_{nj}) \cdot \text{grad}(R^n S_{nq}) d\Omega$$

from (19), or

$$\sum_{p=-n+1}^{n-1} \underset{\sim}{a}_{pj} \cdot \underset{\sim}{a}_{pq} = n^2 + \int_0^\pi \int_0^{2\pi} \left(\frac{\partial S_{nj}}{\partial \theta} \frac{\partial S_{nq}}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial S_{nj}}{\partial \phi} \frac{\partial S_{nq}}{\partial \phi}\right) \sin\theta \, d\theta d\phi \quad .$$

The double integral can be converted, by integration by parts, to

$$- \int_0^\pi \int_0^{2\pi} S_{nj} \left\{\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S_{nq}}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 S_{nq}}{\partial \phi^2}\right\} \sin\theta \, d\theta d\phi$$

when the boundedness and periodicity of the spherical harmonic are borne in mind. But, from (17),

$$n(n+1)S_{nj} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S_{nj}}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 S_{nj}}{\partial \phi^2} = 0 \quad .$$

Hence

$$\sum_{p=-n+1}^{n-1} \underset{\sim}{a}_{pj} \cdot \underset{\sim}{a}_{pq} = n(1+2n)\delta_{jq} \quad (j,q = -n,\ldots,n) \quad . \tag{A.5}$$

It may be shown in a similar way from (A.2) that

$$\sum_{q=-n}^{n} \underset{\sim}{b}_{qk} \cdot \underset{\sim}{b}_{qp} = n(2n-1)\delta_{kp} \quad (p,k = -n+1,\ldots,n-1) \quad . \tag{A.6}$$

A further result of some interest can be derived.

$$\sum_{p=-n+1}^{n-1} \sum_{j=-n}^{n} \{(2n-1)\underset{\sim}{a}_{pj} + (2n+1)\underset{\sim}{b}_{jp}\}\cdot\{(2n-1)\underset{\sim}{a}_{pj} + (2n+1)\underset{\sim}{b}_{jp}\}$$

$$= \sum_{j=-n}^{n} (2n-1)\{(2n-1)n(1+2n) + (2n+1)n(1-2n)\}$$

$$+ (2n+1) \sum_{p=-n+1}^{n-1} \sum_{j=-n}^{n} \underset{\sim}{b}_{jp} \cdot \{(2n-1)a_{pj} + (2n+1)\underset{\sim}{b}_{jp}\}$$

by (A.5) and (A.3). The first sum on the right clearly vanishes and so does the second when (A.4) and (A.6) are invoked. Therefore

$$(1-2n)\underset{\sim}{a}_{pj} = (1+2n)\underset{\sim}{b}_{jp} \quad (p = -n+1,\ldots,n-1;\ j = -n,\ldots,n) \quad . \tag{A.7}$$

If (A.7) is incorporated in (A.5) and (A.6) we obtain

$$\sum_{p=-n+1}^{n-1} \underset{\sim}{b}_{jp} \cdot \underset{\sim}{b}_{qp} = \frac{n(1-2n)^2}{1+2n} \delta_{jq} \quad (j,q = -n,\ldots,n) \quad , \tag{A.8}$$

$$\sum_{q=-n}^{n} \underset{\sim}{a}_{kq} \cdot \underset{\sim}{a}_{pq} = \frac{n(1+2n)^2}{2n-1} \delta_{kp} \quad (p,k = -n+1,\ldots,n-1) \quad . \tag{A.9}$$

APPENDIX B

In this appendix some properties of the field in T_+ will be obtained when T_+ is a homogeneous medium (with λ_0, μ_0, ρ_0 all positive and finite) and the radiation conditions are imposed on the field. Then, it has already been explained in §3 that

$$u_k(\underset{\sim}{x}) = \int_S \{\tau_{ji} g_{jk}(\underset{\sim}{x},\underset{\sim}{y}) n_i - \lambda_0 n_m u_m \frac{\partial}{\partial y_j} g_{jk}(\underset{\sim}{x},\underset{\sim}{y})$$
$$- \mu_0 (n_i u_j + n_j u_i) \frac{\partial}{\partial y_i} g_{jk}(\underset{\sim}{x},\underset{\sim}{y})\} dS_y ,$$

for $\underset{\sim}{x}$ in T_+. Using the explicit formula for g_{jk} in (4) we obtain

$$\underset{\sim}{u} = \text{grad}\, \psi + \text{grad div}\, \underset{\sim}{p} + \omega^2 b^2 \underset{\sim}{p} \tag{B.1}$$

where

$$\psi(\underset{\sim}{x}) = \frac{1}{4\pi\rho_0\omega^2} \frac{\partial}{\partial x_j} \int_S \{n_i \tau_{ij} - \lambda_0 n_m u_m \frac{\partial}{\partial y_j} - \mu_0 (n_i u_j + n_j u_i) \frac{\partial}{\partial y_i}\} \frac{e^{-i\omega a|\underset{\sim}{x}-\underset{\sim}{y}|}}{|\underset{\sim}{x}-\underset{\sim}{y}|} dS_y ,$$

$$p_j(\underset{\sim}{x}) = -\frac{1}{4\pi\rho_0\omega^2} \int_S \{n_i \tau_{ij} - \lambda_0 n_m u_m \frac{\partial}{\partial y_i} - \mu_0 (n_i u_j + n_j u_i) \frac{\partial}{\partial y_i}\} \frac{e^{-i\omega b|\underset{\sim}{x}-\underset{\sim}{y}|}}{|\underset{\sim}{x}-\underset{\sim}{y}|} dS_y .$$

So long as $\underset{\sim}{x}$ keeps away from S

$$\nabla^2 \psi + \omega^2 a^2 \psi = 0 , \tag{B.2}$$

$$\nabla^2 \underset{\sim}{p} + \omega^2 b^2 \underset{\sim}{p} = \underset{\sim}{0} . \tag{B.3}$$

On account of (B.3), (B.1) can be rewritten as

$$\underset{\sim}{u} = \text{grad}\, \psi + \text{curl curl}\, \underset{\sim}{p} . \tag{B.4}$$

Now, if $|\underset{\sim}{x}| > |\underset{\sim}{y}|$, $|\underset{\sim}{x}-\underset{\sim}{y}|^{-1} e^{-i\omega a|\underset{\sim}{x}-\underset{\sim}{y}|+i\omega a|\underset{\sim}{x}|}$ can be expanded in a uniformly and absolutely convergent series of powers of $1/|\underset{\sim}{x}|$ and any number of derivatives can be taken without destroying this property. Also S is bounded so that there is a finite R' such that S is completely enclosed in a sphere with centre at the origin and of radius R'. With R, θ, ϕ the usual spherical polar coordinates, it can now be deduced that, for $R > R'$,

$$\psi(\underset{\sim}{x}) = e^{-i\omega aR} \sum_{n=0}^{\infty} f_n(\theta,\phi)/R^{n+1} \quad , \tag{B.5}$$

$$\mathrm{curl}\ \underset{\sim}{p}(\underset{\sim}{x}) = e^{-i\omega bR} \sum_{n=0}^{\infty} \underset{\sim}{g}_n(\theta,\phi)/R^{n+1} \tag{B.6}$$

the series and their derivatives converging uniformly and absolutely. However, ψ must satisfy (B.2). Inserting (B.5) in (B.2) and equating to zero the coefficients of the various powers of $1/R$ we arrive at

$$2i\omega a(n+1)f_{n+1} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f_n}{\partial\phi^2} + n(n+1)f_n = 0 \ (n=0,1,2,\ldots) \ . \tag{B.7}$$

Dealing with each component of curl $\underset{\sim}{p}$ in the same way we have

$$2i\omega b(n+1)\underset{\sim}{g}_{n+1} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \underset{\sim}{g}_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \underset{\sim}{g}_n}{\partial\phi^2} + n(n+1)\underset{\sim}{g}_n = 0 \ (n = 0,1,2,\ldots) \ . \tag{B.8}$$

In addition, the divergence of curl $\underset{\sim}{p}$ must vanish. To meet this requires

$$\underset{\sim}{g}_0 \cdot \underset{\sim}{i}_1 = 0 \quad , \tag{B.9}$$

$$-i\omega b\underset{\sim}{g}_{n+1} \cdot \underset{\sim}{i}_1 + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ \underset{\sim}{g}_n\cdot\underset{\sim}{i}_2) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\underset{\sim}{g}_n\cdot\underset{\sim}{i}_3) = 0 \ (n\geq 0) \tag{B.10}$$

where $\underset{\sim}{i}_1, \underset{\sim}{i}_2, \underset{\sim}{i}_3$ are unit vectors along the directions R, θ, ϕ increasing respectively.

By means of (B.4) - (B.6) the pertinent expansion of $\underset{\sim}{u}$ in powers of $1/R$ can be written down but explicit details will be omitted here. It is sufficient to note that, if $f_0 \equiv 0$, (B.7) implies that $f_n \equiv 0$ $(n\geq 0)$ because we have assumed that $\omega a \neq 0$. Thus $f_0 \equiv 0$ forces $\psi \equiv 0$ in $R > R'$. Taking advantage of the analytical character of the integral for ψ we infer that $\psi \equiv 0$ in T_+. Similarly, $\underset{\sim}{g}_0 \equiv \underset{\sim}{0}$ implies that curl $\underset{\sim}{p} \equiv \underset{\sim}{0}$ in T_+. Consequently, we reach the important conclusion that, <u>if $f_0 \equiv 0$ and $\underset{\sim}{g}_0 \equiv 0$, $\underset{\sim}{u} \equiv \underset{\sim}{0}$ in T_+.</u>

One useful formula is that

$$\underset{\sim}{u} = -i\omega a\, f_0 \frac{e^{-i\omega aR}}{R} \underset{\sim}{i}_1 - i\omega b \underset{\sim}{i}_1 \wedge \underset{\sim}{g}_0 \frac{e^{-i\omega bR}}{R} + O\left(\frac{1}{R^2}\right) \quad . \tag{B.11}$$

<u>REMARK</u>. In the radiation conditions stated in the text and as used above it is asked that $R\underset{\sim}{u}$ be bounded as $R \to \infty$. This extra requirement is unnecessary since (8) is sufficient by itself to justify all the results that have been obtained. The proof of this assertion will now be set out.

For convenience (8) will be repeated here; it is

$$R\{\hat{x}_m \tau_{jm} + i\omega b\mu_0(u_j - u_m\hat{x}_m\hat{x}_j) + i\omega a(\lambda_0+2\mu_0)\hat{x}_j u_m \hat{x}_m\} \to 0 \quad . \tag{B.12}$$

Multiplication of (B.12) by $\hat{x}_j$ supplies

$$R\{\hat{x}_m \tau_{jm}\hat{x}_j + i\omega a(\lambda_0+2\mu_0)u_m\hat{x}_m\} \to 0 \quad . \tag{B.13}$$

The reality of λ_0, μ_0 and ρ_0 permits the observation that

$$\int_\Omega (n_j\tau_{ij}u_i^* - n_j\tau_{ij}^* u_i)R^2 d\Omega = \int_S (n_j\tau_{ij}u_i^* - n_j\tau_{ij}^* u_i)dS \quad . \tag{B.14}$$

In other words the integral on the left of (B.14) is independent of R although τ_{ij} and u_i are functions of R. Now

$$|\hat{x}_m\tau_{jm} + i\omega b\mu_0(u_j - u_m\hat{x}_m\hat{x}_j) + i\omega a(\lambda_0+2\mu_0)\hat{x}_j u_m\hat{x}_m|^2$$

$$= |\hat{x}_m\tau_{jm}|^2 + (\omega b\mu_0)^2|u_j - u_m\hat{x}_m\hat{x}_j|^2 + \{\omega a(\lambda_0+2\mu_0)\}^2|u_m\hat{x}_m|^2 - i\omega b\mu_0\hat{x}_m(\tau_{jm}u_j^* - \tau_{jm}^* u_j)$$

$$+ i\omega\{a(\lambda_0+2\mu_0) - b\mu_0\}(u_m\tau_{jk}^* - u_m^*\tau_{jk})\hat{x}_j\hat{x}_m\hat{x}_k$$

$$= |\hat{x}_m\tau_{jm}|^2 + (\omega b\mu_0)^2|u_j - u_m\hat{x}_m\hat{x}_j|^2$$

$$+ \frac{\{a(\lambda_0+2\mu_0) - b\mu_0\}}{a(\lambda_0+2\mu_0)}|\hat{x}_m\tau_{jm}\hat{x}_j + i\omega a(\lambda_0+2\mu_0)u_m\hat{x}_m|^2$$

$$- \{1 - b\mu_0/a(\lambda_0+2\mu_0)\}|\hat{x}_m\tau_{jm}\hat{x}_j|^2 + \omega^2 ab\mu_0(\lambda_0+2\mu_0)|u_m\hat{x}_m|^2$$

$$- i\omega b\mu_0\hat{x}_m(\tau_{jm}u_j^* - \tau_{jm}^* u_j) \quad .$$

Incorporation of this in (B.14) and application of (B.12), (B.13) shows that

$$\int_\Omega R^2\{|\hat{x}_m\tau_{jm}|^2 - |\hat{x}_m\tau_{jm}\hat{x}_j|^2 + b\mu_0|\hat{x}_m\tau_{jm}\hat{x}_j|^2/a(\lambda_0+2\mu_0) + (\omega b\mu_0)^2|u_j - u_m\hat{x}_m\hat{x}_j|^2$$

$$+ \omega^2 ab\mu_0(\lambda_0+2\mu_0)|u_m\hat{x}_m|^2\}d\Omega$$

must be finite as $R \to \infty$. Since $|\hat{x}_m \tau_{jm}|^2 \geq |\hat{x}_m \tau_{jm} \hat{x}_j|^2$ this is possible only if

$$\int_\Omega |\hat{x}_m \tau_{jm}(R)\hat{x}_j|^2 d\Omega, \ \int_\Omega |\hat{x}_m \tau_{jm}(R)|^2 d\Omega, \ \int_\Omega |u_j(R) - u_m(R)\hat{x}_m \hat{x}_j|^2 d\Omega, \ \int_\Omega |u_m(R)\hat{x}_m|^2 d\Omega$$

are all $O(1/R^2)$ as $R \to \infty$. Therefore

$$\int_\Omega |u_j(R)|^2 d\Omega \leq 2 \int_\Omega \{|u_j - u_m \hat{x}_m \hat{x}_j|^2 + |u_m \hat{x}_m \hat{x}_j|^2\} d\Omega = O(\frac{1}{R^2}) \quad , \qquad \text{(B.15)}$$

$$\int_\Omega |\hat{x}_m \tau_{jm}(R)|^2 d\Omega = O(1/R^2) \quad . \qquad \text{(B.16)}$$

The order estimates (B.15) and (B.16) are enough to enable us to dispense with the extra radiation conditions since they have been derived entirely on the basis of (8) or (B.12). For the purpose of that further condition was to ensure that there was no contribution from the sphere at infinity in coming to the representation of $\underset{\sim}{u}$ in (B.1). Now, the contribution of the sphere at infinity to $\underset{\sim}{u}(\underset{\sim}{y})$ is given by

$$-\int_\Omega \{n_i \tau_{ij} g_{jk}(\underset{\sim}{x},\underset{\sim}{y}) - \lambda_0 n_m u_m \frac{\partial}{\partial x_j} g_{jk}(\underset{\sim}{x},\underset{\sim}{y}) - \mu_0 (n_i u_j + n_j u_i) \frac{\partial}{\partial x_i} g_{jk}(\underset{\sim}{x},\underset{\sim}{y})\} R^2 d\Omega$$

with $R > |\underset{\sim}{y}|$. On the grounds of the explicit formula (4) g_{jk} can be written

$$g_{jk} = f_{jk} + h_{jk}$$

where f_{jk} covers the terms with $e^{-i\omega a}$ and h_{jk} those with $e^{-i\omega b}$. But, because of the uniform expansions in $1/R$,

$$\frac{\partial f_{jk}}{\partial x_i} = -i\omega a \hat{x}_i f_{jk} + O(\frac{1}{R^2}) \quad ,$$

$$\frac{\partial h_{jk}}{\partial x_i} = -i\omega b \hat{x}_i h_{jk} + O(\frac{1}{R^2}) \quad ,$$

as $R \to \infty$ with $|\underset{\sim}{y}|$ fixed. The integrand under consideration can now be expressed as

$$\{\hat{x}_i\tau_{ij} + i\omega a(\lambda_0+\mu_0)\hat{x}_m u_m \hat{x}_j + i\omega a\mu_0 u_j\}f_{jk}R^2$$

$$+ \{\hat{x}_i\tau_{ij} + i\omega b(\lambda_0+\mu_0)\hat{x}_m u_m \hat{x}_j + i\omega b\mu_0 u_j\}h_{jk}R^2 + O(|\underset{\sim}{u}|) \quad .$$

With the benefit of (9)

$$f_{jk} = \hat{x}_j\hat{x}_k F(\theta,\phi)e^{-i\omega aR}/R + O(1/R^2) \quad ,$$

$$h_{jk} = (\hat{x}_j\hat{x}_k - \delta_{jk})H(\theta,\phi)e^{-i\omega bR}/R + O(1/R^2) \quad .$$

Hence the integrand becomes

$$\{\hat{x}_i\tau_{ij}\hat{x}_j + i\omega a(\lambda_0+2\mu_0)\hat{x}_m u_m\}\hat{x}_k FRe^{-i\omega aR}$$

$$+ \{\hat{x}_i\tau_{ij}\hat{x}_j\hat{x}_k - \hat{x}_i\tau_{ik} - i\omega b\mu_0(u_k - u_j\hat{x}_j\hat{x}_k)\}HRe^{-i\omega bR} + O(|\hat{x}_i\tau_{ij}| + |\underset{\sim}{u}|) \quad .$$

The first term need not be discussed further on account of (B.13). As for the second it may be rewritten as

$$[\{\hat{x}_i\tau_{ij}\hat{x}_j + i\omega a(\lambda_0+2\mu_0)u_m\hat{x}_m\}\hat{x}_k - \hat{x}_i\tau_{ik} - i\omega b\mu_0(u_k - u_j\hat{x}_j\hat{x}_k)$$

$$- i\omega a(\lambda_0+2\mu_0)u_m\hat{x}_m\hat{x}_k]HRe^{-i\omega bR}$$

and its contribution therefore disappears by virtue of (B.12) and (B.13).

Moreover,

$$(\int_\Omega |\underset{\sim}{u}(R)|d\Omega)^2 \leq 4\pi \int_\Omega |\underset{\sim}{u}(R)|^2 d\Omega = O(1/R^2) \quad ,$$

$$(\int_\Omega |\hat{x}_i\tau_{ij}|d\Omega)^2 = O(1/R^2)$$

on account of (B.15) and (B.16). Hence the value of the integral over the large sphere is zero in the limit as $R \to \infty$. Therefore the formula (B.1) has been verified solely under the condition (8). Since the remainder of Appendix B is founded on the representation (B.1) alone its conclusions are still correct without the additional radiation condition. Accordingly, our contention that (8) is a sufficient radiation condition by itself has been vindicated.

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A uniqueness theorem is established for the scattering of harmonic elastic waves by a body with continuously varying parameters placed in a homogeneous medium.

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